

NORMAL MODES IN SYMPLECTIC STRATIFIED SPACES

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ABSTRACT. We generalize the Weinstein-Moser theorem on the existence of nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits near a relative equilibrium in a Hamiltonian system with continuous symmetries.

More specifically we significantly improve a result proved earlier jointly with Tokieda: we remove a strong technical hypothesis on the symmetry group.

1. INTRODUCTION

The goal of the paper is to generalize the Weinstein-Moser theorem ([W1, Mo, W2, MnRS, B] and references therein) on the nonlinear normal modes (i.e., periodic orbits) near an equilibrium in a Hamiltonian system to a theorem on the existence of relative periodic orbits (the normal modes of the title) near relative equilibria of a symmetric Hamiltonian system.

More specifically let (M, ω_M) be a symplectic manifold with a proper Hamiltonian action of a Lie group G and a corresponding moment map $\Phi : M \rightarrow \mathfrak{g}^*$. Assume that the moment map is equivariant. Let $h \in C^\infty(M)^G$ be a G -invariant Hamiltonian. We will refer to the tuple $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$ as a **symmetric Hamiltonian system**.

The main result of the paper is the following theorem (the terms used in the statement are explained below):

Theorem 1. *Let $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$ be a symmetric Hamiltonian system. Suppose $m \in M$ is a relative equilibrium for the system such that the coadjoint orbit through $\mu = \Phi(m)$ is locally closed. Suppose further that there exists a symplectic slice $\Sigma \hookrightarrow M$ through m such that the Hessian $d^2(h|_\Sigma)(m)$ is positive definite.*

Then for every sufficiently small $E > 0$ the set $\{h = E + h(m)\} \cap \Phi^{-1}(\mu)$ (if nonempty) contains a relative periodic orbit of h .

More precisely, let h_μ denotes the reduced Hamiltonian on the reduced space $M_\mu = \Phi^{-1}(G \cdot \mu)/G$. Then for every sufficiently small $E > 0$ the set $\{h_\mu = E + h(m)\}$ contains N or more periodic orbits of the reduced Hamiltonian h_μ , where N is the Liusternik-Schnirelman category of the union of closed strata in the symplectic link of the point $\bar{m} \in M_\mu$ corresponding to $m \in M$.

Recall that given a symmetric Hamiltonian system $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$, a point $m \in M$ is a **relative equilibrium** of the Hamiltonian vector field X_h of h (a relative equilibrium of h for short), if the trajectory of X_h through m lies on the G orbit through m . Equivalently, since the flow of X_h is G equivariant, it descends to a flow on the quotient M/G ; m is a **relative equilibrium** if the corresponding point $\bar{m} \in M/G$ is fixed by the induced flow. Thus if m is a relative equilibrium, then the whole G orbit $G \cdot m$ consists of relative equilibria. Similarly, we say that a trajectory $\gamma(t)$ a G invariant Hamiltonian vector field X_h is a **relative periodic orbit**

Date: February 8, 2008.

Supported by the NSF grant DMS-9803051.

(r.p.o.) if there is $T > 0$ and $g \in G$ such that $\gamma(T) = g \cdot \gamma(0)$. Equivalently $\gamma(t)$ is relatively periodic if the corresponding trajectory $\bar{\gamma}(t)$ in M/G is periodic.

Recall next that if a Lie group G acts properly on a manifold M , then at every point $x \in M$ there is a **slice** for the action of G , i.e., there is a submanifold S passing through x , which is invariant under the action of the isotropy group G_x of x , is transverse to the orbit $G \cdot x$ and such that the open set $G \cdot S$ is diffeomorphic to the associated bundle $G \times_{G_x} S$.

If additionally the manifold M has a G invariant symplectic form ω_M , then the local normal form theorem of Marle and of Guillemin and Sternberg [Ma, GS] guaranties that we can find a symplectic submanifold Σ passing through x which is G_x invariant with the property that the tangent space $T_x \Sigma$ is a maximal symplectic subspace of the tangent space to the slice $T_x S$. Moreover, Σ can be chosen to be G_x equivariantly symplectomorphic to a ball about the origin in a linear symplectic representation of G_x on $T_x \Sigma$. Such a submanifold is called a **symplectic slice**.

The symplectic link of a point \bar{m} in a reduced space M_μ is a symplectic stratified space which is an invariant of a singularity of M_μ at \bar{m} . A precise definition is given later. If the space M_μ is smooth near \bar{m} then the symplectic link of \bar{m} is smooth: it is the complex projective space $\mathbb{C}P^{n-1}$ where $n = \frac{1}{2} \dim M_\mu$.

If the isotropy group of the point m is trivial, then the reduced space M_μ is smooth near the corresponding point \bar{m} . Moreover the symplectic slice Σ through m is symplectomorphic to a neighborhood of \bar{m} in M_μ , and under the identification of Σ with an open subset of the reduced space the restriction $h|_\Sigma$ is the reduced Hamiltonian h_μ . In this case Theorem 1 reduces to a theorem of Weinstein on the existence of nonlinear normal modes of a Hamiltonian system near an equilibrium:

Theorem 2 (Weinstein, [W1]). *Let h be a Hamiltonian on a symplectic vector space V such that the differential of h at the origin $dh(0)$ is zero and the Hessian at the origin $d^2h(0)$ is positive definite. Then for every small $\varepsilon > 0$, the energy level $h^{-1}(h(0) + \varepsilon)$ carries at least $\frac{1}{2} \dim V$ periodic orbits of (the Hamiltonian vector field of) h .*

On the other hand, if m is a *singular* point of the moment map Φ , then the reduced space M_μ at μ is a stratified space, and the reduced dynamics preserves the stratification [AMM, SL, BL]. Unless the stratum through \bar{m} is an isolated point, we have again $\frac{1}{2} \dim(\text{stratum})$ families of r.p.o.'s, provided appropriate conditions hold on the Hessian of the restriction of the reduced Hamiltonian to the stratum. But what if the stratum through \bar{m} is a point?

Recently Tokieda and I showed that in the singular case as in the regular case there are relative periodic orbits near the relative equilibrium provided a certain quadratic form is definite and the isotropy group G_μ of μ is a torus [LTk]. Here as above μ is the value of the moment map on the relative equilibrium. The proof amounted to a reduction to the case where the full symmetry group is a torus followed by computation in ‘good coordinates.’ The computation allowed us to reduce our problem to Weinstein’s nonlinear normal mode theorem.¹ This left open a question:

- Can the assumption on the isotropy group of the value of the moment map at the relative equilibrium be removed?

Theorem 1 answers the question affirmatively. The guiding principle comes from [SL]: a symplectic quotient is locally modeled on a symplectic quotient of a symplectic slice (cf. Theorem 15 in [BL]).

¹The assumption that G_μ is a torus is not essential. For the proof to work it is sufficient to assume that μ is split in the sense of [GLS] and that G_μ splits up to a finite cover: $G_\mu = (G_m \times H)/\Gamma$ where G_m is the isotropy group of the relative equilibrium, H a complementary subgroup and Γ is a finite group.

We end the paper with Proposition 11 which provides a practicable method for checking the existence of a symplectic slice Σ through a relative equilibrium m so that the Hessian $d^2(h|_\Sigma)(m)$ is positive definite.

Acknowledgments. I thank Chris Woodward for a very useful discussion and Yuli Rudiak for patiently answering my e-mail messages.

As I was writing up this paper I discovered that Ortega and Ratiu have independently obtained a similar result [OR2].

2. PROOF OF THEOREM 1

Our first step is to reduce the proof to a special case where the manifold M is a symplectic vector space and the relative equilibrium is an equilibrium. Let $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$ be a symmetric Hamiltonian system and suppose $m \in M$ is a relative equilibrium for the system. Then the restriction h_Σ of h to the symplectic slice Σ through m satisfies $dh_\Sigma(m) = 0$. This is because the Hamiltonian vector field of h points along the the group orbit $G \cdot m$ and the tangent space to the orbit $T_m(G \cdot m)$ lies in the symplectic perpendicular to the symplectic slice directions. Consequently the Hessian $d^2h_\Sigma(m) = d^2(h|_\Sigma)(m)$ is well-defined.

Recall next that if the coadjoint orbit through $\mu = \Phi(m)$ is locally closed, then the reduced space $M_\mu := \Phi^{-1}(G \cdot \mu)/G$ is locally isomorphic, as a symplectic stratified space, to the reduction at zero of a symplectic slice Σ through m by the action of the isotropy group G_m : see, for example [BL, Theorem 15]. Moreover, it follows from the proof of Theorem 15, *op. cit.*, that if h is any G invariant Hamiltonian on M then the corresponding reduced Hamiltonian h_μ on M_μ near $\bar{m} = G \cdot m$ can also be obtained by first restricting h to Σ and then carrying out the reduction by the group G_m .

Since the symplectic slice is equivariantly symplectomorphic to a ball in a symplectic vector space with a linear symplectic action of a compact Lie group, it follows that in order to prove Theorem 1, it suffices to prove the following special case:

Theorem 3. *Let (V, ω_V) be a symplectic vector space with a linear symplectic action of a compact Lie group K , and let $\Phi_V : V \rightarrow \mathfrak{k}^*$ denotes the corresponding homogeneous moment map. Assume that $\Phi_V^{-1}(0) \setminus \{0\}$ is nonempty, i.e., assume that the reduced space $V_0 = \Phi_V^{-1}(0)/K$ is not one point. Suppose $h_V \in C^\infty(V)^K$ is a K invariant Hamiltonian with $dh_V(0) = 0$, and suppose the Hessian $d^2h_V(0)$ is positive definite. Then for every $E > h_V(0)$ sufficiently close to $h_V(0)$ the set $\{h_0 = E\}$ in the reduced space V_0 contains N or more periodic orbit of the reduced Hamiltonian h_0 . Here N is the Liusternik-Schnirelman category of the union of closed strata in the symplectic link of the point $* \in V_0$ corresponding to $0 \in V$.*

The idea of the proof of Theorem 3 is straightforward. Consider the quadratic part $q(x)$ of the Hamiltonian h_V at zero, that is, let $q(x) = d^2h(0)(x, x)$. Let q_0 denote the corresponding reduced Hamiltonian on reduced space V_0 . We will see that for every sufficiently small $E > 0$ and for certain strata \mathcal{T} of V_0 the manifolds

$$\{q_0 = E\} \cap \mathcal{T}$$

contain weakly nondegenerate periodic manifolds $C \subset \{q_0 = E\} \cap \mathcal{T}$ of q_0 . Then by a theorem of Weinstein [W2, p. 247], every such *compact* manifold C gives rise to $\text{Cat}(C/S^1)$ periodic orbits of the reduced Hamiltonian h_0 in the stratum \mathcal{T} , where Cat denotes the Liusternik-Schnirelman category.

Recall a characterization of weakly nondegenerate periodic manifolds given in a corollary on p. 246 of [W2], which we take as our definition.

Definition 4. Let (N, ω_N, h) be a Hamiltonian system. A submanifold C of N consisting of periodic orbits of the Hamiltonian vector field X_h of h is **weakly nondegenerate** iff for each orbit c in C

- (i) $X_h(c) \neq 0$ and
- (ii) the space $\{x \in T_{c(0)}(h^{-1}(E)) \mid x - Px \text{ is a multiple of } X_h(c(0))\}$ has the same dimension as C . Here $E = h(c(0))$, and $P : T_{c(0)}(h^{-1}(E)) \rightarrow T_{c(0)}(h^{-1}(E))$ denotes the linearization of the Poincaré map along the periodic orbit c .

Thus to prove Theorem 3 it is enough to establish the existence of compact weakly nondegenerate periodic manifolds of the reduced Hessian q_0 and to estimate the Liusternik-Schnirelman category of the quotients of these manifolds by S^1 .

We now proceed with a proof of Theorem 3. Since the function q is quadratic, its Hamiltonian vector field X_q is linear, hence of the form $X_q(x) = \xi x$ for some linear map $\xi \in \mathfrak{sp}(V, \omega)$, the Lie algebra of the symplectic group $\text{Sp}(V, \omega)$. Since q is definite, ξ must lie in a compact Lie subalgebra of $\mathfrak{sp}(V, \omega)$; in particular the closure of $\{\exp t\xi \mid t \in \mathbb{R}\}$ is a torus $T \subset \text{Sp}(V, \omega)$. Since q is K -invariant, the groups K and T commute in $\text{Sp}(V, \omega)$. Since both groups are compact, we may assume that $V = \mathbb{C}^n$ ($n = \frac{1}{2} \dim V$), that K is a subgroup of $U(n)$ and that T is contained in the standard maximal torus of $U(n)$.

Lemma 5. *Let $(V, \omega, K, \Phi : V \rightarrow \mathfrak{k}^*, q(x) \in C^\infty(V)^K)$ be as above. Then the Hamiltonian q has no relative equilibria in the set $\Phi^{-1}(0) \setminus \{0\}$. Consequently the action of the torus T generated by q on the reduced space V_0 has no fixed points in $V_0 \setminus \{*\} = (\Phi^{-1}(0) \setminus \{0\})/K$.*

Proof. If $v \in \Phi^{-1}(0) \setminus \{0\}$ is a relative equilibrium then

$$d\langle \Phi, \eta \rangle(v) = dq(v)$$

for some $\eta \in \mathfrak{k}$. Since Φ is quadratic homogeneous, we have $tv \in \Phi^{-1}(0)$ for all $t > 0$. Hence $v \in \ker d\langle \Phi, \eta \rangle(v)$. On the other hand, since q is definite, the ray $\{tv \mid t > 0\}$ is transverse to the level set $\{q = q(v)\}$, hence $v \notin \ker dq(v)$. Contradiction. \square

The structure of the symplectic quotient V_0 . Next we tersely recall a number of results explained in [SL]. Suppose as above that T is a subtorus of the maximal torus of $U(n)$ and that K is a closed subgroup of $U(n)$ which commutes with T . Let U denote the central circle subgroup of $U(n)$. Then the actions of U and K on \mathbb{C}^n commute, the action of U is Hamiltonian and a moment map f for the action of U on \mathbb{C}^n can be taken to be $f(z) = \|z\|^2$.

Since by assumption $\Phi^{-1}(0) \setminus \{0\} \neq \emptyset$, the group U is not contained in K . In fact the Lie algebras of U and K intersect trivially. Let us prove this. If $\mathfrak{u} \cap \mathfrak{k} \neq 0$ then, since $\dim \mathfrak{u} = 1$ we would have $\mathfrak{u} \subset \mathfrak{k}$. But then $\|z\|^2$ would be a component of the K -moment map Φ and so $\Phi^{-1}(0)$ would only contain zero.

Since Φ is homogeneous, the level set $\Phi^{-1}(0)$ is a cone on $\Phi^{-1}(0) \cap S^{2n-1}$ where S^{2n-1} is the standard round sphere in \mathbb{C}^n of radius 1, $S^{2n-1} = \{z \mid \|z\|^2 = 1\}$. Hence the reduced space V_0 is a cone on the set $L := (\Phi^{-1}(0) \cap S^{2n-1})/K$, i.e., $V_0 = \overset{\circ}{c}(L) := (L \times [0, \infty))/\sim$ where $(x, 0) \sim (x', 0)$ for all $x, x' \in L$. The vertex $*$ of the cone corresponds to $0 \in V$. Moreover, (see [SL, Corollary 6.12]) the set L is a stratified space; it is the link of the singularity of V_0 at $*$.² The stratifications of L and of V_0 are related: given a stratum \mathcal{S} of L , the set $\mathcal{S} \times (0, \infty) \subset \overset{\circ}{c}(L)$ is a stratum of V_0 .³

²Calling L the link is slightly nonstandard. Strictly speaking we should call L the link only if the set $\{*\}$ is a stratum; that is, if the set of fixed points V^K is only the origin.

³There is one exception: if $V^K \neq \{0\}$ then $(V^K \cap S^{2n-1})/K = V^K \cap S^{2n-1}$ is a stratum of L , but $(V^K \cap S^{2n-1}) \times (0, \infty) = V^K \setminus \{0\}$, rather than V^K which is a stratum of V_0 . See previous footnote.

By [SL, Theorem 5.3] the action of the circle U on L is locally free and preserves the stratification of L . The quotient $\mathcal{L} := L/U$ is again a stratified space. In fact, \mathcal{L} is a symplectic stratified space since it is a reduction of \mathbb{C}^n by the action of $K \times U$. The space \mathcal{L} is called the **symplectic link** of $*$ in the symplectic stratified space V_0 .

Remark 6. The symplectic link \mathcal{L} has two natural decompositions. There is a stratification of \mathcal{L} into manifolds as a symplectic stratified space. There is also a coarser decomposition: since the action of U on L is locally free and preserves the stratification of L , the quotients of the strata of L form a decomposition of the symplectic link \mathcal{L} into symplectic orbifolds: $\mathcal{L} = \coprod_{\mathcal{S} \subset L} \pi(\mathcal{S})$ where \mathcal{S} are strata of L and $\pi : L \rightarrow \mathcal{L}$ is the U -orbit map. We will use the coarser decomposition.

Remark 7. Note that since the Hamiltonian action of T on V commutes with the action of $U \times K$, it descends to a Hamiltonian action on the symplectic link \mathcal{L} .

Finally recall a description of the symplectic structure on the strata of the reduced space V_0 [SL, Theorem 5.3]: For each stratum \mathcal{S} of L there exists a connection one-form $A_{\mathcal{S}}$ on the Seifert fibration $\mathcal{S} \rightarrow \mathcal{S}/U$ such that the curvature of $A_{\mathcal{S}}$ is a symplectic form. The reduced symplectic form on $\mathcal{S} \times (0, \infty)$ is $d(sA_{\mathcal{S}})$, where s denotes the natural coordinate on $(0, \infty)$. There is no loss of generality in assuming that the connections $A_{\mathcal{S}}$ are T -invariant.

We now study one (connected) stratum P of the link L . Denote by B the quotient of P by the action of U : $B = P/U$. Then $U \rightarrow P \xrightarrow{\pi} B$ is a Seifert fiber bundle. Denote the connection one form on P by A , so that the symplectic form on $P \times (0, \infty)$ is $d(sA)$. By assumption B is a symplectic orbifold.

Lemma 8. *Let $S^1 \rightarrow P \xrightarrow{\pi} B$ be a Seifert fibration over an even dimensional closed orbifold B . Suppose there exists a connection one-form A on P so that the form $d(sA)$ on $P \times (0, \infty)$ is symplectic (s is the coordinate on $(0, \infty)$). Suppose further that a torus T acts on P without fixed points, and that the action of T commutes with the action of S^1 and preserves the connection A . Then the action of T on $(P \times (0, \infty), d(sA))$ is Hamiltonian. Let F denotes a corresponding moment map.*

Then for a generic vector Y in the Lie algebra \mathfrak{t} of T and for any $s_0 \in (0, \infty)$ the set $\pi^{-1}(B^T) \times \{s_0\}$ consists of nondegenerate periodic manifolds of the Hamiltonian $H = \langle F, Y \rangle$ on the energy surfaces $\{H = E\}$ for appropriate E 's.

Proof. Since the action of T preserves the one-form sA , the action of T on $(P \times (0, \infty), d(sA))$ is Hamiltonian. The action of S^1 on $(P \times (0, \infty), d(sA))$ is also Hamiltonian: $f(p, s) = s$ is a corresponding moment map. Consequently $f^{-1}(1)/S^1$ is a symplectic orbifold diffeomorphic to B ; from now on we identify B and $f^{-1}(1)/S^1$.

The Hamiltonian action of T on $(P \times (0, \infty), d(sA))$ descends to Hamiltonian action on B . Since B is compact and the action of T is Hamiltonian, the set of fixed points B^T is nonempty. In fact B^T is a disjoint union of connected symplectic suborbifolds of B (see for example [LT1] for more details on Hamiltonian group actions on orbifolds). For any point $x \in \pi^{-1}(B^T)$, the T orbit $T \cdot x$ is contained in the S^1 orbit $S^1 \cdot x$. Since T acts on P without fixed points we in fact have that $T \cdot x = S^1 \cdot x$ for any $x \in \pi^{-1}(B^T)$. Consequently the union of manifolds $\pi^{-1}(B^T) \times (0, \infty)$ consists of periodic manifolds of the Hamiltonian H .

It remains to check that for a connected component Σ of $\pi^{-1}(B^T)$, the manifold $(\Sigma \times (0, \infty)) \cap \{H = E\}$ is a nondegenerate periodic manifold of H . Now the time t map of the flow of the Hamiltonian vector field of H on $P \times (0, \infty)$ is given by $(p, s) \mapsto ((\exp tY) \cdot p, s)$ where $\exp : \mathfrak{t} \rightarrow T$ is the exponential map.

So let (p, s) be a point in $(\Sigma \times (0, \infty)) \cap \{H = E\}$. Then (p, s) is a relative S^1 equilibrium of H . Hence the differential dH at (p, s) is proportional to the differential of the S^1 moment map, which is ds . Hence $T_{(p,s)}\{H = E\} = T_p P$. Therefore it's enough to compute the differential at p of the “Poincaré map” $P \rightarrow P$, $q \mapsto \exp(\tau Y) \cdot q$, where τ is the smallest positive number with $\exp(\tau Y) \cdot p = p$.

Since the T orbit of p is a circle, the isotropy group of p is of the form $\Gamma \times T_2$, where Γ is a finite abelian subgroup of T and T_2 is a subtorus of T of codimension one. Moreover, we can split T as $T = T_1 \times T_2$ where T_1 is isomorphic to S^1 and contains Γ .

Let us next assume, to make the exposition simpler, that Γ is trivial. Then it follows from the slice theorem that a neighborhood of p in P is T equivariantly diffeomorphic to a neighborhood of $(1, 0, 0)$ in $T_1 \times \mathbb{C}^m \times \mathbb{C}^k$ where $m = \dim \Sigma - 1$. Here $T = T_1 \times T_2$ acts on $T_1 \times \mathbb{C}^m \times \mathbb{C}^k$ by

$$(\lambda, t) \cdot (\mu, w_1, \dots, w_m, z_1, \dots, z_k) = (\lambda\mu, w_1, \dots, w_m, \chi_1(t)z_1, \dots, \chi_k(t)z_k),$$

where $\chi_1, \dots, \chi_k : T_2 \rightarrow U(1)$ are nontrivial characters of T_2 .

Let $pr_\alpha : T \rightarrow T_\alpha$, $\alpha = 1, 2$ denote the projections. Then $pr_1(\exp(\tau Y)) = 1$. We claim that for all r between 1 and k , $\chi_r(pr_2(\exp(\tau Y)))$ is of the form $e^{2\pi i y_r}$ where y_r are **irrational** numbers. Note that the claim implies immediately that the algebraic multiplicity of the eigenvalue 1 of the differential of the “Poincaré map” $q \mapsto \exp(\tau Y) \cdot q$ is $\dim \Sigma$, hence that $(\Sigma \times (0, \infty)) \cap \{H = E\}$ is a nondegenerate periodic manifold of H .

The claim holds because the one-parameter subgroup $\{\exp(tY) \mid t \in \mathbb{R}\}$ is dense in T . More specifically, let e_1, \dots, e_s be a basis of the integral lattice of the torus T which is compatible with the splitting $T = T_1 \times T_2$ (so that $\{\exp(te_1) \mid t \in \mathbb{R}\} = T_1$ and e_2, \dots, e_s is a basis of the integral lattice of T_2). Then $Y = a_1 e_1 + \sum_{j=2}^s a_j e_j$ for some $a_j \in \mathbb{R}$. Moreover, since the one-parameter subgroup defined by Y is dense in T , the sum $\sum_{j=1}^s q_j a_j$ is not a rational number for any s tuple of rational numbers (q_1, \dots, q_s) . Since $\exp(pr_1(\tau Y)) = 1$, $a_1 = \pm \frac{1}{\tau}$. Consequently

$$\chi_r(pr_2(\exp(\tau Y))) = \chi_r\left(\pm \sum_{j=2}^s \frac{a_j}{a_1} e_j\right) = e^{2\pi i (\pm \sum_{j=2}^s \frac{d\chi_r(e_j)}{a_1} a_j)}$$

and the claim follows (note that $d\chi_r(e_j)$ are integers).

If the group Γ is not trivial, then a neighborhood of p in P is modeled by the quotient $(T_1 \times \mathbb{C}^m \times \mathbb{C}^k)/\Gamma$, where Γ acts on T^1 by multiplication and on $\mathbb{C}^m \times \mathbb{C}^k$ linearly by $m + k$ characters, so that the actions of T and Γ commute. The same argument as above still works: for the Poincaré map on the quotient to have an eigenvector in \mathbb{C}^k with eigenvalue 1 it is necessary for $\chi_r(pr_2(\exp(\tau Y)))$ to be a root of unity. But this is impossible as we have seen. This proves Lemma 8. \square

In fact in proving Lemma 8, we have proved more:

Proposition 9. *Let (V, ω) be a symplectic vector space with a linear action of a compact Lie group K and a corresponding homogeneous moment map $\Phi : V \rightarrow \mathfrak{k}^*$. Let $q \in C^\infty(V)^K$ be a K invariant positive definite quadratic Hamiltonian; let q_0 be the corresponding reduced Hamiltonian on $V_0 = \Phi^{-1}(0)/K$. Let $T \subset \text{Sp}(V, \omega)$ be the torus generated by q . Let L be the link of the point $* \in V_0$ corresponding to 0, let \mathcal{L} be the symplectic link and let $\pi : L \rightarrow \mathcal{L}$ be the orbit map.*

Then for every stratum \mathcal{S} of the link L such that the fixed point set $\pi(\mathcal{S})^T$ is nonempty and for every $E > 0$ the manifold

$$\{q_0 = E\} \cap (\mathcal{S} \times (0, \infty))$$

contains a weakly nondegenerate periodic manifold C of q_0 : $C = \pi^{-1}(\pi(\mathcal{S})^T)$. If the orbifold $\pi(\mathcal{S})^T$ is compact, then the manifold C is compact as well.

Note that by construction the circle action on the periodic manifolds C in Proposition 9 is simply the action of the circle U . Hence $C/S^1 = \pi^{-1}(\pi(\mathcal{S})^T)/U = \pi(\mathcal{S})^T$. It follows from a result of Weinstein [W2, p. 247] that in Theorem 3 the number of periodic orbits of the Hamiltonian h_0 on a given energy surface $\{h_0 = E\}$ is bounded below by

$$(2.1) \quad N_1 = \sum \text{Cat}(\pi(\mathcal{S})^T)$$

where the sum is taken over all strata \mathcal{S} of the link L such that the sets $\pi(\mathcal{S})^T$ are **compact**. Since the link L is compact, the closed strata of L must be compact. It follows that the number N_1 in equation (2.1) is positive.

The bound given by (2.1) is somewhat unsatisfactory — it ultimately depends on the Hamiltonian, while no such dependence is present in Weinstein’s nonlinear normal modes theorem (Theorem 2 above). We will see in Lemma 10 below that $\text{Cat}(\pi(\mathcal{S})^T) \geq \text{Cat}(\pi(\mathcal{S}))$ for any closed stratum \mathcal{S} of the link L . Consequently

$$N_1 \geq \sum \text{Cat}(\pi(\mathcal{S})),$$

where the sum is taken over all closed strata \mathcal{S} of the link L . This will finish our proof of Theorem 3 hence of Theorem 1.

Lemma 10. *Let B be a closed symplectic orbifold with a Hamiltonian action of a torus T . Then the Liusternik-Schnirelman category $\text{Cat}(B^T)$ of the set of T -fixed points is bounded below by the Liusternik-Schnirelman category of B : $\text{Cat}(B^T) \geq \text{Cat}(B)$.*

Proof. We use open sets in our definition of the category. Let $f : B \rightarrow \mathbb{R}$ be a generic component of the moment map for the action of T on B so that B^T is precisely the set of critical points of f . The function f is Bott-Morse. Therefore B decomposes as a disjoint union of the unstable orbifolds W_1, \dots, W_k of the gradient flow of f . Clearly $\text{Cat}(\coprod W_k) = \text{Cat}(B^T)$. Now “thicken” W_j ’s by replacing them with their tubular neighborhoods \tilde{W}_j inside B . Then $\text{Cat}(\tilde{W}_j) = \text{Cat}(W_j)$, the sets \tilde{W}_j ’s are open and $\cup_j \tilde{W}_j = B$. Hence $\text{Cat}(B) \leq \sum \text{Cat}(\tilde{W}_j) = \text{Cat}(B^T)$. \square

We end the paper by describing a practicable method for checking the existence of a symplectic slice Σ through a relative equilibrium m of a symmetric Hamiltonian system $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$ so that the Hessian $d^2(h|_\Sigma)(m)$ is positive definite.

Recall that if a point m is a relative equilibrium of a symmetric Hamiltonian system, then there exists a vector $\eta \in \mathfrak{g}$ so that

$$(2.2) \quad d(h - \langle \Phi, \eta \rangle)(m) = 0.$$

Then the Hessian $d^2(h - \langle \Phi, \eta \rangle)(m)$ is a well-defined quadratic form, which we will use shortly. The vector η is not unique: for every ζ in the Lie algebra \mathfrak{g}_m of the isotropy group of m , the vector $\eta + \zeta$ also satisfies $d(h - \langle \Phi, \eta + \zeta \rangle)(m) = 0$. It is not hard to show that η has to lie in the isotropy Lie algebra \mathfrak{g}_μ where $\mu = \Phi(m)$.

Since by assumption the action of G on M is proper, the isotropy group G_m is compact. Hence we can choose a G_m invariant inner product on the Lie algebra \mathfrak{g} and use it to define an orthogonal complement \mathfrak{m} of \mathfrak{g}_μ in \mathfrak{g}_μ . There exists a *unique* vector $\eta \in \mathfrak{m}$ so that (2.2) holds. The vector η is called the **orthogonal velocity** of the relative equilibrium m [OR1].

Proposition 11. *Let m be a relative equilibrium of a symmetric Hamiltonian system $(M, \omega_M, \Phi : M \rightarrow \mathfrak{g}^*, h \in C^\infty(M)^G)$ and let $\eta \in \mathfrak{g}_\mu$ be the orthogonal velocity of m with respect to some G_m invariant inner product on \mathfrak{g} (where $\mu = \Phi(m)$). Suppose that the quadratic form $d^2(h - \langle \Phi, \eta \rangle)(m)|_{\ker d\Phi(m)}$ is semi-definite of maximal possible rank (the dimension of a symplectic slice at m). Then there exists a symplectic slice Σ through m such that the form $d^2(h|_\Sigma)(m)$ is positive definite.*

Proof. The proof is a standard computation that uses the local normal form of the moment map of Marle and of Guillemin and Sternberg [Ma, GS]. Similar computations are carried out in [LS, p. 1643] and in [OR1]. We use the version of the normal form theorem described in [BL, pp. 211–215] which we now recall without proofs:

Let the symbols (M, ω) , G , $\Phi : M \rightarrow \mathfrak{g}^*$, $\mu = \Phi(m)$, \mathfrak{g}_m , \mathfrak{g}_μ and \mathfrak{m} have the same meaning as above.

The null directions of the restriction $\omega(m)|_{\ker d\Phi(m)}$ is $T_m(G_\mu \cdot m)$. Hence $V = \ker d\Phi(m)/T_m(G_\mu \cdot m)$ is naturally a symplectic vector space. Denote the corresponding symplectic form by ω_V . Moreover, the linear action of the isotropy group G_m on $\ker d\Phi(m)$ descends to a linear symplectic action on (V, ω_V) . Denote the corresponding homogeneous moment map by Φ_V .

The G_m invariant inner product on \mathfrak{g} chosen above defines G_m equivariant embeddings: $i : \mathfrak{g}_m^* \rightarrow \mathfrak{g}^*$ and $j : \mathfrak{m}^* \rightarrow \mathfrak{g}^*$. Note that by construction of i and \mathfrak{m} we have that $\langle i(\ell), \eta \rangle = 0$ for any $\ell \in \mathfrak{g}_m^*$ and any $\eta \in \mathfrak{m}$.

There exists a closed two-form σ on the associated bundle $Y = G \times_{G_m} (\mathfrak{m}^* \times V)$ and an open G equivariant embedding ψ of a neighborhood the zero section of $Y \rightarrow G/G_m$ into M with the following properties.

1. $\psi([1, 0, 0]) = m$.
2. $\psi^*\omega = \sigma$.
3. $(\psi^*\Phi)([g, \lambda, v]) = \text{Ad}^\dagger(g)(\mu + j(\lambda) + i(\Phi_V(v)))$ for all $[g, \lambda, v] \in G \times_{G_m} (\mathfrak{m}^* \times V)$.
4. The embedding $\iota : (V, \omega_V) \hookrightarrow (G \times_{G_m} (\mathfrak{m}^* \times V), \sigma)$, $\iota(v) = [1, 0, v]$ is symplectic. Consequently for a small enough neighborhood U of 0 in V , $\psi(\iota(U))$ is a symplectic slice through m .

We now prove that $\Sigma = \psi(\iota(U))$ is the desired symplectic slice. Since $d(h - \langle \Phi, \eta \rangle)(m) = 0$, the Hessian $d^2(h - \langle \Phi, \eta \rangle)(m)$ is well-defined and behaves well under restrictions. In particular, $d^2(h - \langle \Phi, \eta \rangle)(m)|_{T_m(G_\mu \cdot m)} = d^2(h - \langle \Phi, \eta \rangle)|_{G_\mu \cdot m}(m)$. Since h is G invariant and since for any $a \in G_\mu$ we have $\langle \Phi, \eta \rangle(a \cdot m) = \langle \text{Ad}^\dagger(a)\Phi(m), \eta \rangle = \langle \text{Ad}^\dagger(a)\mu, \eta \rangle = \langle \mu, \eta \rangle = \langle \Phi, \eta \rangle(m)$. Therefore $(h - \langle \Phi, \eta \rangle)|_{G_\mu \cdot m}$ is constant and hence $d^2(h - \langle \Phi, \eta \rangle)(m)|_{T_m(G_\mu \cdot m)} = 0$.

Since the null directions of $\omega(m)|_{\ker d\Phi(m)}$ is $T_m(G_\mu \cdot m)$, it follows that for any symplectic slice Σ' through m which is tangent to $\ker d\Phi(m)$, we have

$$T_m\Sigma' \oplus T_m(G_\mu \cdot m) = \ker d\Phi(m).$$

Combining this with the previous computation we see that the rank of $d^2(h - \langle \Phi, \eta \rangle)(m)|_{\ker d\Phi(m)}$ is at most $\dim \Sigma'$. Thus by assumption $d^2(h - \langle \Phi, \eta \rangle)(m)|_{T_m\Sigma'}$ is positive definite for any symplectic slice which is tangent to $\ker d\Phi(m)$. It is easy to check that the manifold $\psi(\iota(U))$ is such a slice. We finally show that

$$d^2(h - \langle \Phi, \eta \rangle)(m)|_{\psi(\iota(U))} = d^2(h|_{\psi(\iota(U))})(m).$$

Now for any $u \in U$ we have $\langle \Phi, \eta(\psi(\iota(u))) \rangle = \langle \psi^*\Phi, \eta \rangle([1, 0, u]) = \langle \mu + j(0) + i(\Phi_V(u)), \eta \rangle = \langle \mu, \eta \rangle$ (since $\langle i(\mathfrak{g}_m^*), \eta \rangle = 0$ by construction of i and η). Therefore $d^2(h - \langle \Phi, \eta \rangle)(m)|_{\psi(\iota(U))} = d^2((h - \langle \Phi, \eta \rangle)|_{\psi(\iota(U))})(m) = d^2(h|_{\psi(\iota(U))} - \langle \mu, \eta \rangle)(m) = d^2(h|_{\psi(\iota(U))})(m)$. \square

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